Descending motions in viscous liquids

How can one describe / model / predict / understand various descending motion patterns related to

- Falling leaves and snowflakes
- Falling business cards and paper strips
- Spread of seeds from trees as they fall
- Flying insects and birds
- Descending airplanes

Amazingly, there is still no satisfactory simple physical explanation of the observed motions.

Descending motion of a ball

Galileo Galilei hypothesize: "A body would fall with a strictly uniform acceleration, as long as the resistance of the medium through which it was falling remained negligible."

How did he discover this? What would he observed if he really dropped balls of the same material, but different masses, from the Leaning Tower of Pisa?

What really happens with a heavy ball dropped to fall in still air?

The air-resistance force increases approximately proportionally to the square of the speed and reaches the magnitude of the the weight force. After that, the body falls with a constant terminal velocity.

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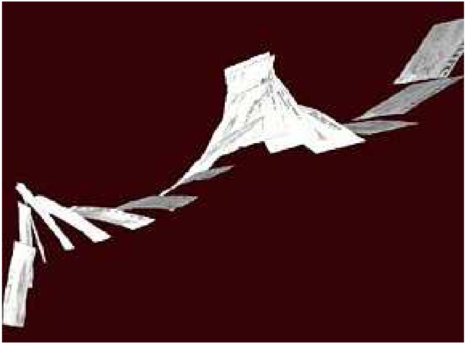
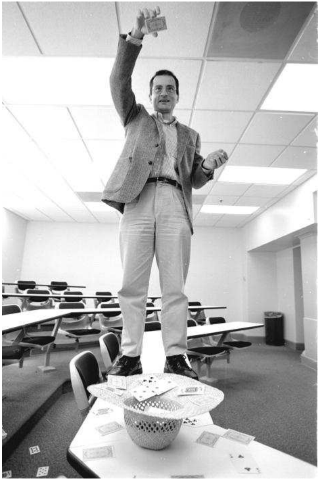
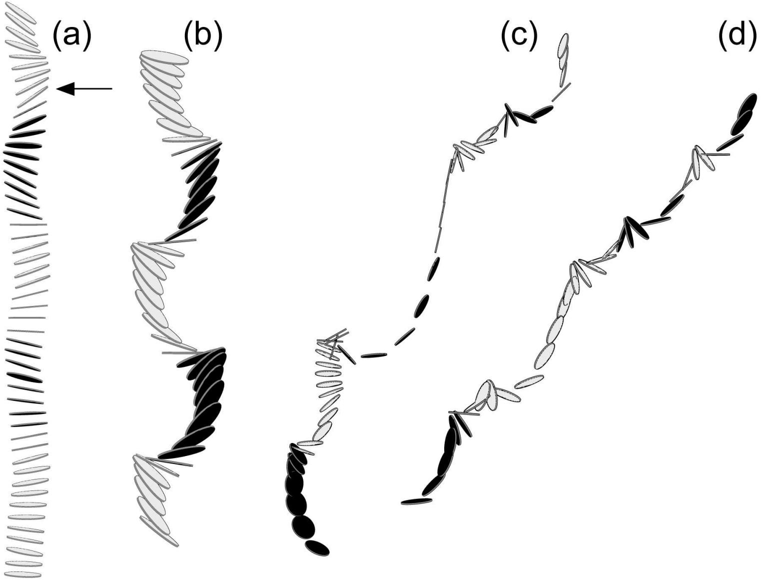
Experiments

When the body is lighter and has a more complicated shape, other aerodynamic forces are created by the air and they influence the motion!

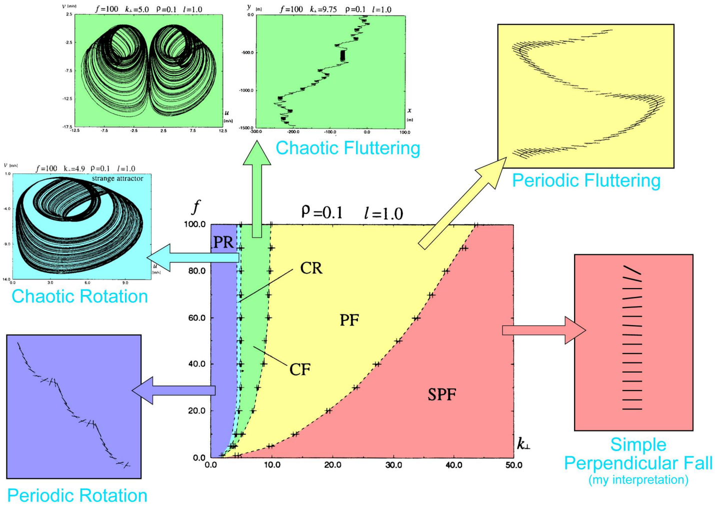




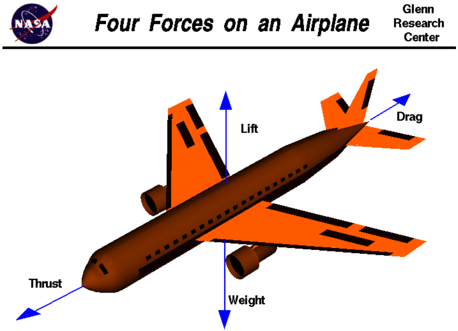
Some of possible falling patterns



If the body is shaped as a thin plate, there are 5 possible descending patterns:



Forces in the air, the case of steady flow



- Lift: the force orthogonal to the direction of the motion
- Drag: the resistance force directed against the motion
- Weight: the vertical gravitational force
- Added mass effect: the air is partially glued to the object effectively changing its mass / buoyant (Archimedes) force

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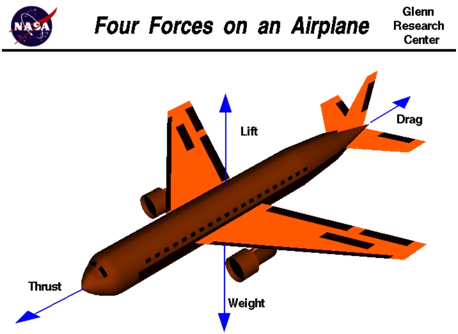

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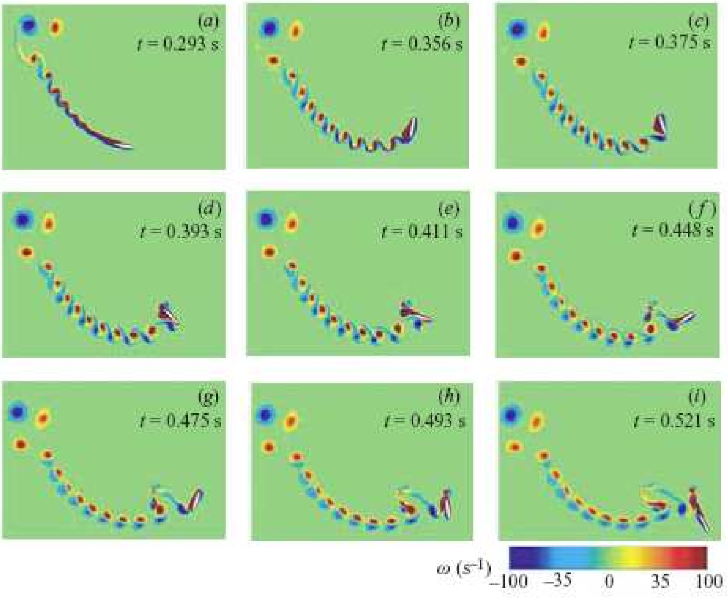
The flow is quit mysterious...



Possible Theoretical Modeling Approaches

- Solving Full Navier-Stokes equations:
  - Computationally very expensive
  - Sharp corners are not allowed
  - How to extract relevant physics?
- Deriving Exact Models with Simplified Assumptions:
  - Possible to obtain analytic results if the model is simple enough!
  - Seems to be impossible to take into account viscosity, gravity, etc.
- Obtaining Phenomenological Models:
  - Approximate lift, drag, etc. phenomenologically
  - Can include all relevant forces!
  - **How to justify that the model is adequate?**

Model by Andersen, Pesavento, Wang (2005)

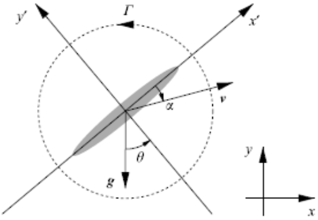


FIGURE 13. The velocity components $v_{x'}$ and $v_{y'}$ in the laboratory reference frame are defined with respect to the coordinate system following the rotation of the body, whereas v_x and v_y are the horizontal and the vertical velocity component in the laboratory reference frame, respectively. The angle of attack, α, satisfies $\alpha \in [-\pi/2, \pi/2]$ and it is negative in the example sketched.

$$
\begin{aligned}
(m + m_{11})\dot{v}_{x'} &= (m + m_{22})\,\dot{\theta}\,v_{y'} + \mathrm{Lift}_{x'} - m'\,g\,\sin(\theta) + \mathrm{Drag}_{x'}\,,\\
(m + m_{22})\,\dot{v}_{y'} &= -(m + m_{11})\,\dot{\theta}\,v_{x'} + \mathrm{Lift}_{y'} - m'\,g\,\cos(\theta) + \mathrm{Drag}_{y'}\,,\\
(J + J_a)\,\ddot{\theta} &= (m_{11} - m_{22})\,v_{x'}\,v_{y'} + \mathrm{Lift}_{\tau} + \mathrm{Drag}_{\tau}\,,
\end{aligned}
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where $m = \rho_s hl$, $m' = (\rho_s - \rho_f)hl$, and $J = \rho_s hl(l^2 + h^2)/12$.

Model by Andersen, Pesavento, Wang (2005), [cont'd]

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- Assumption 1: the **added masses** can be computed as for the steady (turbulence-free) flow:

$$m_{11} \approx \frac{\pi}{4}\rho_f h^2, \quad m_{22} \approx \frac{\pi}{4}\rho_f l^2, \quad J_a \approx \frac{\pi}{128}\rho_f (l^2 - h^2)^2.$$

- Assumption 2: **Lift** (aerodynamic force orthogonal to the motion) and **Drag** (aerodynamic force parallel to the motion) **depend on velocities only** (as in the case of steady flow with small angles of attack).

To complete the model, Lift and Drag can be approximated along the steady-state motion experimentally via curve fitting!

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Validation of the model

By construction, the experimentally recorded steady-state behavior approximately satisfies the differential equations. However, to validate the model, the following must be verified:

- Is it possible to recover the other observable motions?
- Are all these motions attract trajectories initiated on a reasonable distance — exponential orbital stability & regions of attractions.
- Qualitative changes in behavior are not implied by reasonable variations in parameters — robustness with respect to physical parameters.
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Validation procedure

Let us introduce the following notation

- Δp_1 is a vector of small deviations in the values of the physical parameters.
- Δp_2 is a vector of small parameters defining a family of possible mismatches in the experiment-based descriptions of the Drag and the Lift.

We have developed a computational procedure for derivation of a dynamic description for time evolutions of the 5 elements of a vector $\zeta(t)$ defining deviations of the 6 states $(x, y, v_{x'}, v_{x'}, \theta, \dot{\theta})$ of the system from the nominal orbit in the form:

$$\frac{d}{dt}\zeta(t) = A(t, \Delta p_1, \Delta p_2)\,\zeta(t) + B(t, \Delta p_1, \Delta p_2)$$

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Outcome for the validation procedure

The following can be deduced analyzing the equation

$$\frac{d}{dt}\zeta(t) = A\big(t, \Delta p_1, \Delta p_2\big)\,\zeta(t) + B\big(t, \Delta p_1, \Delta p_2\big)$$

- If $B\big(t, 0, 0\big) \equiv 0$, the target periodic solution exists.
- If the transition matrix for $A\big(t, 0, 0\big)$ is Hurwitz, the target periodic solution is orbitally exponentially stable.
- If the solutions stay in a vicinity of the origin, for **sufficiently small** Δp_1, we have robustness with respect to physical parameters.
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